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USING SPACE AND OPERATORS IN A FIXED POINT APPROACH TO TRAFFIC FLOW DYNAMICS

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Abstract

In this paper, build a unique mathematical framework that combines nonlinear operator theory with quasi-partial metric spaces to model traffic flow dynamics. Our method naturally captures asymmetric travel delays and congestion effects, in contrast to traditional models based on Euclidean or Banach spaces. We provide a traffic evolution operator on a quasi-partial metric space and use fixed point theory to prove the presence and distinctiveness of Stability. There is discussion of applications related to intelligent transportation systems and network congestion.

Keywords: *Nonlinear Operator Theory; Quasi-Partial Metric Spaces; Traffic Flow Dynamics; Fixed Point Theory.*

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1. Introduction

Traffic flow modeling traditionally relies on Traffic flow modeling traditionally relies on PDE models, variational formulations and Network optimization. But Real traffic is nonlocal and asymmetric, Travel delay is not equal to symmetric distance and Classical metric spaces are insufficient. Recent studies show traffic systems can be modeled in nonlinear metric or measure spaces

Traditional traffic flow modeling [1,2,6-10] has long relied on well-established mathematical frameworks such as the Lighthill–Whitham–Richards model and the Aw–Rascle–Zhang model, along with vibrational formulations and network optimization techniques. These approaches are typically grounded in classical continuum mechanics and assume that traffic dynamics can be described through local interactions and symmetric spatial relationships. In particular, such models are built upon the structure of standard metric spaces, where distance is symmetric and satisfies strict axioms like the triangle inequality and zero self-distance. While these assumptions simplify analysis and allow elegant mathematical formulations, they do not fully reflect the complexity of real-world traffic systems. In practice, traffic behavior exhibits strongly nonlocal and asymmetric characteristics. Drivers respond not only to immediate neighboring vehicles but also to conditions further ahead, making the system inherently nonlocal.

Moreover, travel time between two points is often direction-dependent due to factors such as one-way roads, varying traffic signals, and congestion patterns; thus, the notion of distance becomes asymmetric. Additionally, delays such as waiting at intersections imply that the "distance" from a point to itself may not be zero, contradicting classical metric assumptions. These features highlight the inadequacy of traditional metric spaces in capturing realistic traffic dynamics.

To address these limitations, more generalized mathematical structures such as Quasi-metric space and Quasi-partial metric space [16,17] provide a more suitable framework. These spaces relax the symmetry and self-distance constraints, allowing a more accurate representation of travel delays and directional dependencies. When combined with concepts from Operator Theory, traffic evolution can be modeled as a nonlinear operator acting on a space of traffic states. This perspective enables the study of equilibrium conditions through fixed-point analysis, where steady traffic patterns correspond to fixed points of the operator. Furthermore, incorporating nonlocal interactions via integral terms leads to more realistic models that account for long-range driver responses.

Overall, the shift from classical metric-based models to frameworks involving quasi-partial metrics, operator theory, and nonlocal dynamics represents a significant advancement in traffic flow modeling. It not only resolves fundamental inconsistencies between theory and real-world behavior but also opens new avenues for rigorous mathematical analysis and practical applications in transportation systems.

Now we explain Asymmetry (Directional Traffic Delay):

Let X_1 denote a traffic network's locations (or nodes), and let $q1: X_1 \times X_1 \to [0,\infty)$

Represent the travel delay function. Then, in general,

$q1(x,y) \neq q1(y,x), \forall\ x,y \in X_1,\ x \neq y.$

The directional nature of traffic flow is expressed in this inequality, where the delay from x to y varies from the delay from y to x. The concept of quasi-metric space readily captures this kind of asymmetry [3,4,5] which is a basic feature of actual traffic systems, by allowing for different distances or delays between two points depending on the direction of travel.

2. Preliminaries

2.2 Quasi-Partial Metric Space

A pair $(X_1,q1)$such that:

a) $q1(x,x) \geq 0$
b) $q1(x,x) \leq q1(x,y)$
c) $q1(x,y) \neq q1(y,x)$
d) $q1(x,z) \leq q1(x,y) + q1(y,z) - q1(y,y)$
e) $q1\ (y,y) \neq 0$ which shows congestion representation
f) $q1\ (x,y) \neq q1\ (y,x)$implies directional traffic delay

2.3 Operator Theory

Let T_{LR}:$X_1 \rightarrow X_1$. $T_{LR}(x)$ updated traffic state can represent density evolution and delay propagation .Traffic modeling already uses delay operators in networks

Traffic Operator Representation:

Define a nonlinear operator [17] and let X_1 be an adequate space of traffic states, including density functions. Let T_{LR}:$X_1 \rightarrow X_1$, the latest version of traffic flow after one development step is shown by the image corresponding to T_{LR} (x) or $T_{LR}\rho(x)$ when working with densities for each $x \in X_1$. The dynamics of traffic systems [3,4,15] can be explained in a unified way using this operator concept.

Furthermore, the operator T_{LR} can be developed to symbolize the following:

- **Density evolution**, where T_{LR} updates vehicle density based on flow dynamics and interactions;
- **Delay propagation**, where travel times are modified according to congestion and network conditions.

Thus, traffic evolution can be interpreted as an iterative process $X_{1n+1}= T_{LR}(x_n)$,

and fixed points ,satisfy $T_{LR}(x)=x$ represent steady-state traffic arrangements. Such formulations naturally arise in Operator Theory [14] and show that modern traffic modeling already implicitly employs delay-type operators on networks.

3. Traffic Flow Model in QPMS

Let $X_L$denotes the set of all locations (or nodes) in a traffic network, and let

q1: $X_L \times X_L \rightarrow [0,\infty)$ be a function that shows how long it takes to get from one place to another. The axioms of a metric space are assumed to be satisfied by this function in classical models, however real traffic systems defy these assumptions because of asymmetry and intrinsic delays. We model the system in a quasi-partial metric space in order to address this.

Definitions:

Traffic Delay Structure

The pair (X_1,q1) is called a quasi-partial metric space if for all $x,y,z \in X_1$, the following assertions hold:

1. Non-negativity:$q1\ (x,y) \geq 0$.
2. Self-distance (non-zero allowed):$q1\ (x,x) \geq 0$

This represents waiting time or signal delay at location x.

3. Asymmetry:$q1\ (x,y) \neq q1\ (y,x)$. This captures directional traffic delay.
4. Modified Triangle Inequality: $q1\ (x,z) \leq q1\ (x,y) + q1\ (y,z) - q1\ (y,y)$

This reflects the cumulative nature of delays while accounting for intermediate waiting times [18,20].

Traffic State Representation

The traffic density function is given by$\mathcal{D}_\rho$:$X_1 \rightarrow [0,\infty)$, where $\mathcal{D}_\rho(x)$ is the vehicle density at x.

.Now, define a nonlinear operator

T_{LR}: $X_1 \rightarrow X_1$, so that the updated traffic condition at point x is represented by (T_{LR} $\mathcal{D}_\rho$)(x).

Operator-Based Traffic Evolution

One of the ways to modeltraffic evolution is given by the following:

$\mathcal{D}_{\rho n+1}(x) = (T_{LR}\, \mathcal{D}_\rho)(x)$.

Where the operator T_{LR}included

1. Nonlocal interactions (vehicles reacting to downstream traffic),
2. Delay effects via the quasi-partial metric q1 (x,y),
3. Network constraints such as signals and one-way flows.

This framework is naturally studied using Operator Theory.

4. Traffic Evolution Operator

Nonlinear operatordefined by:

$T_{LR}(x)(h)=ax(h)+(1-a)F(x)(h)\ T_{LR}(x)(h)$

Where F(x): flow update function and a∈ (0, 1)

5. Main Theorem

Theorem:

Let (X_1,q1) be a complete q1uasi-partial metric space and T_{LR}satisfy

$q1(T_{LR}x, T_{LR}y) \leq \upsilon q1(x,y), 0<\upsilon< 1$

ThenT_{LR}has a uniq1ue fixed point, Traffic system reaches stable equilibrium

Proof :

Construct sequence $x_{n+1}= T_{LR}(x_n)$Show contraction$q1(x_n, x_{n+1})\rightarrow 0$

Now we prove $\{x_n\}$ is Cauchy sequence and if it is Complete then it is converge

Where Limit satisfies$T_{LR}(x^*)=x^*$

Let(X_1,q1) be a complete quasi-partial metric space and let$F=T_{LR}:X_1\rightarrow X_1$satisfy the contraction condition

$q1(Fx,Fy)\leq \upsilon\ q1(x,y),\ 0 < \upsilon < 1\ ,\forall x,y\in X_1$.

$q1(Fx,Fy)\leq \upsilon\ q1(x,y),\ 0 < \upsilon < 1$

We prove that $F=T_{LR}$ Fhas a unique fixed point.

Choose an arbitrary point $x_0\in X_1$. Define the sequence $\{x_n\}$ recursively by

$x_{n+1}=F(x_n)= T_{LR}(x_n),n=0,1,2,\ldots$

Now, we have to show that $q1(x_n,x_{n+1})\rightarrow 0$

By using the contractive condition,

$q1(x_{n+1},x_{n+2})=q1(Fx_n,Fx_{n+1})\leq \upsilon q1(x_n,x_{n+1})$.

Which implies, $q1(x_{n+1},x_{n+2}) \leq \upsilon\ q1(x_n,x_{n+1})$.

Iterating this inequality yields $q1(x_n,x_{n+1}).\leq \upsilon n\ q1(x_0,x_1)$.

Since 0< υ<1, we know that $\upsilon n \to 0$ as $n \to \infty$.

Therefore, $q1(x_n,x_{n+1}) \to 0$.

Hence the distance between successive iterates tends to zero.

Again,we prove,$\{x_n\}$ is a Cauchy Sequence

Let p >n. By the inequality of qpm,

$q1(x_n,x_p) \leq q1(x_n,x_{n+1})+q1(x_{n+1},x_{n+2})+\cdots+q1(x_{p-1},x_p)$.

Thus,

$q1(x_n,x_p) \leq \sum_{i=n}^{p-1} q1(x_i,x_{i+1})$

Based on the above approximation, $q1(x_i,x_{i+1}) \leq \upsilon i\ q1(x_0,x_1)$.

We obtain

$q1(x_n,x_p) \leq q1(x_0,x_1) \sum_{i=n}^{p-1} \upsilon^i$

As $\sum_{i=n}^{p-1} \upsilon^i$ is a geometric series which satisfies

$$\sum_{i=n}^{p-1} \upsilon^i = \upsilon^n \frac{1-\upsilon^{(p-n)}}{1-\upsilon} \leq \frac{\upsilon^n}{1-\upsilon}$$

We get,

$q1(x_n,x_p) \leq \frac{\upsilon^n}{1-\upsilon} q1(x_0,x_1)$

As $n \to \infty$, $\frac{\upsilon^n}{1-\upsilon} \to 0$.

Hence,

$q1(x_n,x_p) \to 0$ as $p,n \to \infty$.

Therefore $\{x_n\}$ is a Cauchy sequence in $(X_1,q1)$.

Now, Trough Completeness prove the Convergence Property

Since $(X_1,q1)$ is complete and $\{x_n\}$is Cauchy sequence, there exists some

$x^* \in X_1$.

In such a way that $x_n \to x^*$.

That is, $q1(x_n,x^*) \to 0$ as $n \to \infty$.

Here, Show that the Limit is a Fixed Point

We now prove that,

$T_{LR}(x^*)=x^*$.

Since $x_{n+1}=F(x_n)$, we have

$q1(Fx^*,x_{n+1})=q1(Fx^*,Fx_n)$.

By using the property of contraction,

$q1(Fx^*,Fx_n) \leq \upsilon\ q1(x^*,x_n)$

As $n \to \infty$, $q1(x^*,x_n) \to 0$,

Hence q1(Fx^*,x_{n+1}) 0.

But x_{n+1}→x^*. Therefore, q1 (Fx^*, x^*)=0.

By the property of the quasi-partial metric,Fx^*=x^*.

Therefore,

T_{LR} (x^*)=x^*.

Thus x^* is a fixed point of T_{LR}.

Now prove the uniqueness of the Fixed Point

Suppose $\alpha, \beta \in X_1$are fixed points of F. Then $F\alpha = \alpha$, $F\beta = \beta$.

Bycontractive condition,

q1(α, β)=q1$(F\alpha, F\beta)$≤ υ q1(α, β).This implies

q1(α, β) ≤ υ q1(α, β).

Since 0<α, β< 1, this impliesq1(α, β) = 0.

Hence,$\alpha = \beta$

Therefore the fixed point is unique.

Conclusion

The mapping F = T_{LR}has a unique fixed point $x^* \in X_1$, and the iterative sequence

x_{n+1}=T_{LR}(x_n)converges to x^*. Moreover,

T_{LR}(x^*)=x^*.

6. Application to Traffic Networks

One of the applications explained by Intelligent Transportation Systems (ITS).

Modern smart transportation systems use traffic network models alongside sensors, GPS data, and artificial intelligence algorithms for real-time monitoring.

Some applications of IT system given by Integrated traffic light, applications include adaptive routing that anticipates congestion and coordinates autonomous vehicles.

Non-local interactions are crucial because vehicles respond to traffic conditions farther ahead in addition to those in their immediate vicinity.

Intelligent Transportation Systems (ITS)

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7. Conclusion

This framework integrates network structure, delay-based geometry, and operator dynamics into a unified model. It provides a mathematically consistent and realistic representation of traffic networks, capable of capturing both control mechanisms and complex congestion behavior

8. Discussion

Advantages of traffic network are models asymmetric delay, captures self-congestion and works beyond linear spaces for example metric-space traffic modeling and operator-based delay systems.

Example 1.

Traffic signal at a busy intersection

Suppose a vehicle travels from point A ie. my home to point B ie. SSTC, Bhilai.

Let

$X=\{\text{traffic states on the road}\}$.

For two traffic states $x,y \in X$, define

$q(x,y)=$ required time for x to y.

and $q(x,x)>0$.

A vehicle may still encounter a waiting delay due to a red light or congestion if the traffic situation has not altered.

suppose,

- x: traffic state at 10:00 AM
- y: traffic state at 10:07 AM

When the signal is red and vehicles are waiting.

$q(x,x)=2.30$ minutes,

Because even in the same traffic state includes an average 2.30 -minute waiting delay.

Similarly,

$q(x, y)=9.30$ minutes

It may represent the delay required for the traffic condition at10:00 AM to evolve into the traffic condition at 10:07 AM.

Traffic slowdown is always in a direction.

$q(x,y)\neq q(y,x)$

Means the delay from x to y need not equal the delay from y to x:

Here we give an example,

q(normal traffic, congested traffic)=9.30 min,

While

q(congested traffic, normal traffic)=17 min.

Since congestion does not disappear immediately so the second transition takes longer

Therefore, the structure becomes The second transition takes longer

$q(x,y)\neq q(y,x)$,

So, a quasi-partial metric is more suitable for a traffic-delay model.

Operator interpretation

Let, a traffic-update operator be $T: X\rightarrow X$. Where $T(x)$ = traffic condition after one signal cycle.

If x represents moderate traffic, $T(x)$ may represent heavy traffic after a red-light cycle.

Repeated application gives $x,T(x),T^2(x),T^3(x),\ldots$ over successive time intervals. Then $T(x*)=x*$

Real-life interpretation

Mathematical concept	Traffic interpretation
x,y	Different traffic conditions
$q(x,y)$	Delay/time to move from condition x to y
$q(x,x)>0$	Delay exists even in the same traffic condition
$q(x,y)$□$=q(y,x)$	Directional/asymmetric traffic delay
$T(x)$	Updated traffic condition
$T(x)=x$	Stable traffic condition
$q(x,y)$	Congestion propagation/travel delay

Conclusion: A quasi-partial metric space is well suited to traffic modeling because delays may remain positive even when the traffic state does not change, and the delay from one state to another may differ from the delay in the reverse direction. Therefore, it provides a natural mathematical framework for modeling waiting times, congestion propagation, and asymmetric travel delays

Now, A simple daily-life example of a Traffic Delay Structure in a Quasi-Partial Metric Space (QPMS) is a traffic signal/intersection during peak hours.

Example: Traffic delay at a traffic signal

Suppose we consider three traffic states:

- x_1: road is almost empty
- y_1: minimal traffic
- z_1: heavy traffic

Let X be the set of all possible traffic states. Define a function

$q(x_1,y_1)$=delay experienced when traffic changes from state $x_1$to state y_1.

In a **quasi-partial metric**, the distance can be asymmetric:

$q(x_1, y_1)\ )\neq q(y_1, x_1)$.

This is realistic because the delay from one traffic state to another need not be the same in the reverse direction.

Numerical example

Consider the following delay matrix:

From/To	x_1	y_1	z_1
x_1	1	3	7
y_1	2	3	9
z_1	4	6	12

For example

$q(x_1, y_1)= 3, q(y_1, x_1)=2.$

Thus,

$q(x_1, y_1))\neq q(y_1, x_1).$

This asymmetry has a natural interpretation: going from an empty road to moderate traffic may create a different delay than going from moderate traffic back to an empty road.

Self-distance

A particularly useful feature of a quasi-partial metric is that

$q(x_1, x_1)$

does not necessarily have to be zero.

For example, for the heavy-traffic state z_1,

$q(z_1, z_1)=12$ minutes.

This can represent the **delay caused by maintaining the same congested state**—for example; vehicles continue waiting at a red signal even though the traffic state remains "heavy."

This is useful in traffic modeling because a traffic state can have an inherent delay or congestion cost.

Traffic interpretation

Suppose a vehicle moves through an intersection:

Empty → Moderate → Heavy.

The corresponding delays could be

$q(x_1, y_1)=3, q(y_1, z_1)=9.$

The total transition effect can be studied through a quasi-partial triangle-type inequality such as

$q(x_1, z_1) \leq q(x_1, y_1) + q(y_1, z_1) - q(y_1, y_1),$

depending on the particular definition of the quasi-partial metric being used.

For instance, if

$q(x_1, z_1)=7, q(x_1, y_1)=3, q(y_1, z_1)=9, q(y_1, y_1)=3,$

So the traffic states satisfy the required generalized triangle relation.

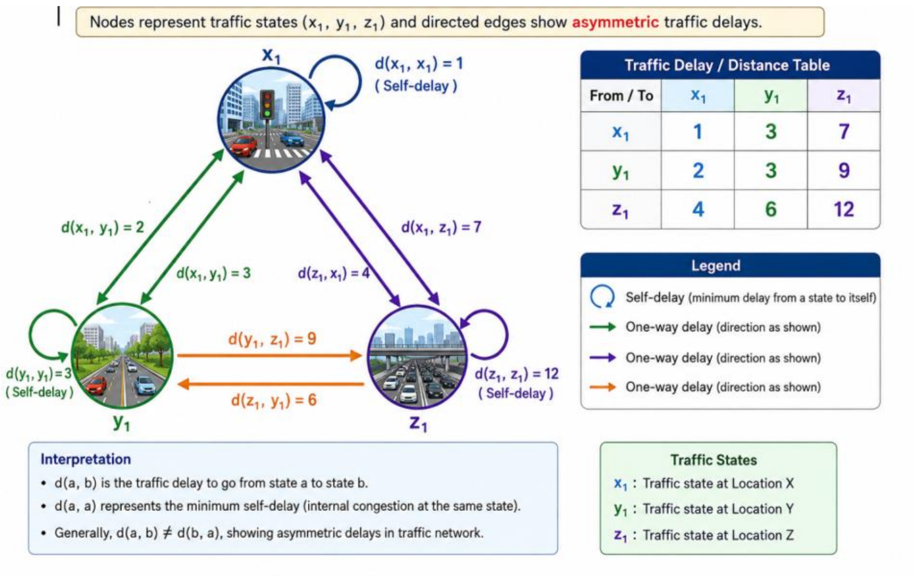

Figure 1. Traffic delay structure in Quasi Partial Metric Sapce

Why this is useful

This example demonstrates three important characteristics:

1. Asymmetry

   $q(x_1, y_1)\ )\neq q\ (y_1, x_1)$,

   representing directional traffic delays.

2. Non-zero self-distance

   $q(x_1, x_1)>0$,

   representing the delay or congestion inherent in maintaining a traffic state.

3. Generalized triangle property

   $q(x_1, z_1) \leq q(x_1, y_1) + q(y_1, z_1) - q(y_1, y_1)$,

   allowing intermediate traffic states to influence the total delay.

Hence, a traffic intersection can be modeled as a quasi-partial metric space, where points represent traffic states and $q(x_1, y_1)$ represents the directional delay between those states.

Conclusion:

A quasi-partial metric provides a mathematical framework for describing traffic situations in which the delay from traffic state x to y can differ from the delay from y to x, while a traffic state itself may have a non-zero inherent delay.

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