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A STUDY OF FIXED-POINT THEOREMS: BANACH'S CONTRACTION PRINCIPLE AND BROUWER'S FIXED POINT THEOREM

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Abstract

The conditions under which a mapping T has at least one point x that maps to itself—that is, T(x)=x—are studied by fixed point theory. These points naturally arise in numerical iteration, optimisation, economics, dynamical systems, and nonlinear, differential, and integral equations. The Banach contraction principle in complete metric spaces, existence, uniqueness, and error estimates are the topics of this assignment. The topological fixed-point theorem of Brouwer, which affirms existence for continuous self-maps of compact convex subsets of finite-dimensional Euclidean space but typically does not declare uniqueness or provide an iterative construction, is then contrasted with Banach's constructive theorem.

Keywords: *Fixed Point Theory, Banach Contraction Principle, Brouwer Fixed-Point Theorem, Complete Metric Spaces, Existence and Uniqueness, Convergence, Error Estimates, Applications.*

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1. Introduction

A mapping's fixed point is a value that the mapping does not change. Despite being simple, the equation T(x)=x encodes a number of important issues. The solution of an integral equation may be a fixed point of an associated integral operator; the equilibrium of a dynamical system is a fixed point of its time evolution map; and a nonlinear equation F(x)=0 can be recast as x=T(x). Equation solving is not the only use for fixed point theorems. They develop structural presumptions, such as completeness, contraction, compactness, convexity, and continuity, that guarantee the presence of solutions. The strengths of these theorems vary; some just demonstrate existence, while others demonstrate uniqueness and provide an explicit convergent approximation approach.

2. Basic Definitions

Let T: X → X, and let X be a nonempty set. A sequence (xn) in a metric space (Xd) converges to x if d(xn,x) tends to 0. A point x^*, X is considered a fixed point of T if T $(x^*) = x^*$. If each Cauchy sequence converges to a point in space X, then that space is complete.

A contraction is a mapping $T:(X_d) \rightarrow (X_d)$ if there is a constant q such that $0 \leq q < 1$.

$d(Tx,Ty) \leq q\, d(x,y)$ for all $x,y \in X$.

The quantity q is referred to as a Lipschitz constant or contraction constant. Strict distance reduction is the main characteristic. A nonexpansive mapping, on the other hand, just fulfils $d(T_x,T_y) \leq d(x,y)$, which is typically insufficient for the Banach conclusion without other assumptions.

3. Banach Contraction Principle

Banach fixed-point theorem. Let $T:X \rightarrow X$ be a contraction with constant $q<1$, and let (X,d) be a nonempty full metric space. Then, x*. x is a unique fixed point of T. Additionally, the iterates $x_{n+1}=T(x_n)$ converge to x* for each starting point x_0 x.

3.1 Proof of Existence

Fix $x_0 \in X$ and define $x_n=T^n(x_0)$. By the contraction inequality,

$d(x_{n+1},x_n) \leq q^n\, d(x_1,x_0)$.

For m>n, the triangle inequality gives

$d(x_m,x_n) \leq (q^n + q^{n+1} + \cdots + q^{m-1})d(x_1,x_0) \leq q^n/(1-q) \cdot d(x_1,x_0)$.

The right-hand side tends to zero as $n \rightarrow \infty$; hence (x_n) is Cauchy. Completeness supplies $x^* \in X$ with $x_n \rightarrow x^*$. Since a contraction is Lipschitz continuous, T is continuous. Therefore $T(x^*)=T(\lim x_n)=\lim T(x_n)=\lim x_{n+1}=x^*$, so x* is a fixed point.

3.2 Proof of Uniqueness

If x* and y* are fixed points, then

$d(x^*,y^*) = d(Tx^*,Ty^*) \leq qd(x^*,y^*)$.

Because $0 \leq q < 1$, this implies $(1-q)d(x^*,y^*) \leq 0$. Since a metric is nonnegative, $d(x^*,y^*)=0$ and $x^*=y^*$. Thus the fixed point is unique.

3.3 Error Estimates

Banach's theorem is constructive. It gives the a priori estimate

$d(x_n,x^*) \leq q^n/(1-q) \cdot d(x_1,x_0)$,

and the a posteriori estimate

$d(x_n,x^*) \leq q/(1-q) \cdot d(x_n,x_{n-1})$.

These bounds give numerical fixed-point iteration stopping conditions. Convergence is fast if q is small, but it can be slow if q is near 1, even if it is still guaranteed.

4. Banach's Theorem

4.1 Affine Mapping on R

Let T:R→R be defined by T(x)=0.4x+3. Then

$|T(x)-T(y)|=0.4|x-y|$.

T is therefore a contraction on the entire metric space R. Since x=0.4x+3 is satisfied by its unique fixed point, x*=5. The iteration $x_{n+1}=0.4x_n+3$ converges to 5 from any x_0.

4.2 A Nonlinear Example

Define T(x)=cos(x)/2 on X=[0,1]. The mean value theorem produces $|T(x)-T(y)|\leq(1/2)|x-y|$ since $|T'(x)|=|\sin(x)|/2\leq 1/2$. Furthermore, T(X) →[cos(1)/2,1/2] →[0,1]. As a result, the iterative technique $x_{n+1}=\cos(x_n)/2$ converges to T's unique fixed point in [0,1].

4.3 The Significance of Completeness

Let T(x)=x/2 and X=(0,1) with the standard metric. The map maps X into X and is a contraction with q=1/2. 0 is the only fixed point that might exist, but 0 →X. As a result, T has no fixed point in X. The reason for the failure is that (0,1) is incomplete. This illustration demonstrates that the completeness assumption cannot be disregarded.

5. Brouwer Fixed-Point Theorem

The Brouwer Theorem. There is at least one fixed point in every continuous mapping from a nonempty compact convex subset K of R2 onto itself. Specifically, there is a fixed point for any continuous map from the closed unit ball B^2 into itself. Banach's theorem and Brouwer's theorem differ significantly. Instead of a contraction requirement, it assumes continuity and a finite-dimensional compact-convex domain. Existence, not uniqueness, is its conclusion. It is topological rather than metric in nature, and its typical proofs make use of degree theory, homology, and Sperner's lemma.

Table 1. Comparison of Banach and Brouwer Fixed-Point Theorems

Feature	**Banach theorem**	**Brouwer theorem**
Setting	Complete metric space	Compact convex subset of R^n
Hypothesis	Contraction	Continuity
Conclusion	Existence and uniqueness	Existence
Construction	Iteration converges	Generally nonconstructive
Typical use	Numerical/analytic equations	Topological existence results

5.1 Illustration

Assume that $T: [0,1] \to [0,1]$ is continuous. In one dimension, Brouwer's theorem suggests that T has a fixed point. Define $g(x)=T(x)-x$, in fact. The intermediate value theory yields x^* with $g(x^*)=0$ since $g(0)=T(0)\geq 0$ and $g(1)=T(1)-1\leq 0$. The comparable conclusion is more deeper in higher dimensions.

6. Applications

6.1 Nonlinear Equations

Select a map T such that $F(x)=0$ is equal to $x=T(x)$ in order to solve $F(x)=0$. Banach's theorem establishes a unique solution and verifies the iteration $x_{n+1}=T(x_n)$ if T is a contraction on a complete invariant set. Selecting a reformulation with a contraction constant smaller than one is the practical challenge.

6.2 Ordinary Differential Equations

For the initial-value problem $y'=f(t,y)$, $y(t_0)=y_0$, integration gives

$$y(t)=y_0 + \int_{t_0}^{t} f(s,y(s))\, ds.$$

An operator on a suitable space of continuous functions is defined by this equation. That operator turns into a contraction on a sufficiently small time interval and under a Lipschitz condition in y. The Picard–Lindelöf theorem's existence and uniqueness components are then obtained using Banach's theorem.

6.3 Integral Equations

For an equation $x(t)=g(t)+\int_a^b K(t,s,x(s))ds$, one studies the induced integral operator T. If the kernel satisfies a suitable Lipschitz bound and the resulting operator norm is below one, T is contractive on an appropriate complete function space. The fixed point is the desired solution.

6.4 Economics and Game Theory

Equilibria, or states where the choice or response rule leaves the state unaltered, are known as fixed points. When continuity and convex feasible sets are involved, Brouwer-type theorems offer existence arguments in the lack of a contraction property (and hence uniqueness).

7. Results

The Banach principle has numerous generalisations. The Schauder fixed-point theorem replaces contraction with continuity plus compactness in suitable Banach-space situations. The Kakutani fixed-point theorem, which is essential to equilibrium theory, generalises the idea to set-valued correspondences. The Knaster--Tarski theorem characterises fixed points by the order structure and deals with monotone mappings on full lattices.

There is a trade-off with these generalisations. Weakening hypotheses typically results in the loss of an efficient iterative process or the originality of the conclusion. When you wish to show existence both rigorously and computationally, the Banach theorem is especially helpful.

8. Limitations and Common Errors

- A continuous self-map need not be a contraction. Continuity alone is not a reason to invoke Banach's theorem.
- A contraction must map the selected set X to itself. Invariance is not sufficient as a test of the inequality. The contraction constant should be strictly less than one. A map with Lipschitz constant 1 need not be covered.
- Completeness is a property of the metric space being used. Closed intervals in R are complete; open intervals need not be.
- Brouwer's theorem guarantees the existence of at least one fixed point, but does not necessarily guarantee uniqueness or convergence of a simple iteration.

9. Conclusion

A common framework for existence, uniqueness, and approximation issues is provided by the theory of fixed points. A contraction has a unique fixed point on a complete metric space, and the iterates converge to it with explicit error control, according to Banach's forceful contraction principle. Brouwer's theorem is more comprehensive in another way: it provides existence from compact convex geometry and continuity, but not from contraction. These findings provide a common paradigm that leads to strong conclusions for nonlinear issues using metric, analytic, and topological structures.

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